Sign of refractive index and group velocity in left-handed media

A. L. Efros and A. L. Pokrovsky

University of Utah, Salt Lake City UT, 84112 USA

Abstract

We argue that the widely spread opinion that the left-handed media (LHM) are characterized by a negative refractive index n_- is misleading. Since n does not enter into Maxwell's equations and boundary conditions, any medium may be described by both positive n and negative $n_- = -n$. Two thermodynamic inequalities are presented, that make a difference between the LHM and the regular media (RM). The first one reads that the group velocity is positive in the RM and negative in the LHM. The second one is that the product $\mathrm{Re}(n)\mathrm{Im}(n)$ is positive in the RM and negative in the LHM. Both inequalities are invariant with respect to the change $n \to n_-$. However, to use n_- one should change some traditional electrodynamics definitions.

INTRODUCTION

Victor Veselago was the first to consider the left-handed media (LHM) [1, 2], which he defined as the media with simultaneously negative and almost real electric permittivity ϵ and magnetic permeability μ in some frequency range. He showed that the LHM have a number of peculiar properties. All these properties follow from the fact that vectors $\mathbf{k}, \mathbf{E}, \mathbf{H}$ of the plane electromagnetic wave form a left-handed rather than a right-handed set. Then, the Poynting vector $\mathbf{S}$ has a direction opposite to the direction of the wave vector $\mathbf{k}$. The left-handed set can be easily obtained from Maxwell's equations for harmonic fields $c\mathbf{k} \times \mathbf{H} = -\omega\epsilon(\omega)\mathbf{E}$ and $c\mathbf{k} \times \mathbf{E} = \omega\mu(\omega)\mathbf{H}$.

One of the most interesting properties of the LHM is a negative refraction at the interface of a regular medium (RM) and a LHM. The negative refraction follows immediately from the continuity of the tangential component of $\mathbf{k}$ and normal component of $\mathbf{S}$. The anomalous Snell's law which describes negative refraction of the wave incident from the RM has a form

$$\frac{\sin\theta_i}{\sin\theta_r} = -\frac{n_l}{n_r}, \tag{1}$$

where θ_r, θ_i are the angles of refraction and incidence respectively, n_l, n_r are positive refractive indices of the LHM and the RM. The negative refraction has been recently observed by Shelby, Smith, and Shultz[3] and by Kosaka[4] in completely different systems.

Veselago was the first to propose that the refractive index n_l in the LHM is negative. In our understanding his only reason was that with this assumption Snell's law Eq.(1) acquires the standard form. Smith and Kroll (SK) argued[5] that in the LHM the refractive index should be negative while the group velocity (GV) is positive. This point of view is widely spread now [6, 7, 8, 9, 10]. Based upon this statement Valanju *et al.*[7] challenged the Veselago's results and claimed that the refraction on the RM-LHM interface is positive.

In this paper we critically analyze this point of view. First we discuss the definition of the refractive index in an isotropic medium. It follows from the wave equation that

$$\omega^2 n^2 = c^2 k^2, \tag{2}$$

where $n^2 = \epsilon\mu$. Now one should relate the wave vector $\mathbf{k}$ to n. The main principle should be that a refractive index represents a property of an isotropic medium and it is independent

of the direction of vector $\mathbf{k}$. We write the relation in the form

$$\mathbf{k} = \frac{\omega n}{c}\mathbf{l}, \tag{3}$$

where $\mathbf{l}$ is a unit vector in the direction of Re$\mathbf{k}$. By definition Re$n = n' > 0$. Note, that in the isotropic RM the vector Im$\mathbf{k}$ has the same direction as Re$\mathbf{k}$. Thus, the above definition gives that in the RM $n' > 0$ and $n'' = \mathrm{Im}(n) > 0$. The definition of Eq.(3) is used in Eq.(83.13) of Landau and Lifshitz textbook[11]. We show, however, that in the LHM Eq.(3) leads to a negative n''.

Another definition, though non-traditional, is as good as the previous one for both types of materials. It reads

$$\mathbf{k} = -\frac{\omega n_-}{c}\mathbf{l}. \tag{4}$$

In this case $n'_- < 0$. We argue here that for both types of materials all results obtained using definition of Eq.(3) may be rewritten through n_- by the change $n \to -n_-$. It follows from the fact that neither Maxwell's equation nor the boundary conditions include the refractive index.

There is the third definition which goes back to Brillouin[12]. One may consider $n = \sqrt{\mu\epsilon}$ as an analytical function of the complex frequency ω in the upper half plane of ω. According to Brillouin one should choose the Riemann surface of this function as a surface where $n \to 1$ at $\omega \to \infty$ along the real axis. Brillouin has shown how to make an analytical continuation to find both n' and n'' everywhere along the real axis for the case $\mu = 1$ and $\epsilon(\omega)$ having an anomalous dispersion. In this case n' happens to be positive. Thus, it does not contradict Eq.(3).

SK show that for some specific form of the product $\epsilon(\omega)\mu(\omega)$ the Brillouin's method gives $n' < 0$ just in the region with both $\epsilon < 0, \mu < 0$. Note, that we are not aware of any general theorem that proves this statement for any functions $\epsilon(\omega)$ and $\mu(\omega)$ with usual analytical properties. The result of SK means, that one should use the definition of Eq.(3) for the frequency range where ϵ, μ are positive and Eq.(4) where both ϵ, μ are negative. However, from our point of view, both definitions are equivalent. SK argued also, that one can obtain the correct sign for the energy losses in the LHM only when $n' < 0$. We show below that this statement is incorrect.

GENERAL THEOREMS

First we prove two important theorems which are independent of the above definitions of the refractive index.

Theorem 1: *The group velocity $\partial\omega/\partial\mathbf{k}$ in an isotropic medium is positive in the RM and negative in the LHM.* In fact, this means that vectors $\partial\omega/\partial\mathbf{k}$ and $\mathbf{k}$ are parallel in the RM and antiparallel in the LHM.

Proof: Taking the gradient of $\omega^2 n^2 = c^2 k^2$ in $\mathbf{k}$ space one gets

$$\frac{\partial\omega}{\partial\mathbf{k}} = \frac{2c^2\mathbf{k}}{d\left[\omega^2 n^2\right]/d\omega}. \tag{5}$$

The total time-averaged electromagnetic energy density of the plane wave is [11]

$$\overline{U} = \frac{1}{16\mu\omega}\frac{d\left[\omega^2 n^2\right]}{d\omega}|E|^2. \tag{6}$$

Since the energy density $\overline{U}$ is positive, one gets that

$$\frac{d}{d\omega}\left[\omega^2 n^2\right] < 0 \quad \text{(for the LHM)}, \tag{7}$$

$$\frac{d}{d\omega}\left[\omega^2 n^2\right] > 0 \quad \text{(for the RM)}. \tag{8}$$

Thus, Eqs.(5,7,8) prove the statement. One can see that the criterion which distinguishes between LHM and RM is independent of the sign of n.

Note: Eq.(5) is valid for both definitions of n given by Eqs(3,4). Assuming the definition of Eq.(3), one can restore the usual equation for the GV

$$\frac{\partial\omega}{\partial\mathbf{k}} = \frac{\mathbf{k}}{k}\,\frac{c}{d(n\omega)/d\omega}. \tag{9}$$

However, this equation is not valid at $n < 0$. Assuming the definition of Eq.(4), one gets

$$\frac{\partial\omega}{\partial\mathbf{k}} = -\frac{\mathbf{k}}{k}\,\frac{c}{d(n_{-}\omega)/d\omega}. \tag{10}$$

SK claim that GV is positive in the LHM because they use Eq.(9) with non-traditional negative n. The most important mistake of Valanju *et al.*[7] is of the same origin.

Corollary: *The main statement of Veselago, that the vectors* $\mathbf{S}$ *and* $\mathbf{k}$ *have opposite directions in the LHM, can be obtained from the condition that the energy of the electromagnetic field is positive without using the first order Maxwell's equations.*

Proof: It is easy to show that for a plane wave in an isotropic nearly lossless medium the time averaged Poynting vector is

$$\mathbf{S} = \overline{U}\frac{\partial\omega}{\partial\mathbf{k}}. \tag{11}$$

Thus, using Theorem 1 one can obtain the main result of Veselago: in the LHM the vectors $\mathbf{S}$ and $\mathbf{k}$ have opposite directions. The negative RM-LHM refraction follows from this result and standard boundary conditions at the interface.

Notes: 1. From Eqs.(7,8) follows that the negative sign of the GV and the Veselago results, including the negative refraction at the RM-LHM interface, are independent of the sign of n.

2. In the isotropic medium with small losses the negativeness of the GV might be a more general signature of the LHM than $\epsilon < 0$, $\mu < 0$.

Theorem 2: *The product $n'n''$ is positive in the RM and negative in the LHM.*

Proof: This statement based upon the thermodynamic inequalities for imaginary parts of ϵ and μ in a passive material [11](p. 274)

$$\epsilon'' > 0, \quad \mu'' > 0. \tag{12}$$

The dispersion equations has a form $k^2 = (n' + in'')^2\omega^2/c^2 = (\epsilon' + i\epsilon'')(\mu' + i\mu'')\,\omega^2/c^2$. The imaginary part of this relation reads

$$\epsilon'\mu'' + \mu'\epsilon'' = 2n'n''. \tag{13}$$

Therefore

$$n'n'' > 0 \;\text{ in the RM,} \quad n'n'' < 0 \;\text{ in the LHM.} \tag{14}$$

SIGN OF REFRACTIVE INDEX

Thus, using the fundamental thermodynamic properties we have shown, that the sign of the product $n'n''$ and the sign of the GV are different for the LHM and RM. Both $n'n''$ and the GV are invariant with respect to change of the sign of n and in the rest of our paper we discuss whether or not the sign of n has any physical meaning.

To illustrate our point of view we consider the same problem as SK. They study an infinite sheet of current at $x = 0$, embedded into the LHM (Fig. 1). The surface current density is

$j = j_0 \exp(-i\omega t)$. They solve the equation for the z-component of the electric field $E(x,t)$

$$\frac{\partial^2 E(x)}{\partial x^2} + \frac{\omega^2}{c^2} n^2 E(x) = -\frac{4\pi i \omega}{c^2} \mu j_0 \delta(x) \tag{15}$$

to show that the power radiated by the sheet of current is positive only if the refractive index is negative.

Now we demonstrate that the same condition can be fulfilled without introducing the negative n. Following Veselago we choose the direction of the wave vector $\mathbf{k}$ from the causality condition. In the LHM it should be directed towards the source of the wave, while in the RM it should be directed outward the source. In both, LHM and RM $\mathbf{S}$ should be directed outward the source (Fig.1). Using boundary condition at $x = 0$

$$\left.\frac{\partial E(x)}{\partial x}\right|_{+} - \left.\frac{\partial E(x)}{\partial x}\right|_{-} = -\frac{4\pi i \omega}{c^2} \mu j_0, \tag{16}$$

the electric field can be written in the form

$$E(x,t) = \frac{2\pi\omega}{kc^2} \mu j_0 e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} = \frac{2\pi\omega}{kc^2} \mu j_0 e^{i(-k|x| - \omega t)}, \tag{17}$$

where $\mathbf{k} = k\mathbf{l}$, $\mathbf{l}$ is a unit vector in the direction of Re$\mathbf{k}$ (see Fig.1). Taking into account that $\omega n = ck$, and introducing real and imaginary parts of the refractive index the solution can be written in the following form

$$E(x,t) = \frac{2\pi\omega}{n\kappa c^2} \mu j_0 e^{-(in' - n'')\kappa|x| - i\omega t}, \tag{18}$$

where $\kappa = \omega/c > 0$. From Eq.(18) one can see that n'' should be negative in the LHM to provide the decay of the wave in the direction of the energy propagation. Since we have chosen $n' > 0$ this result follows also from Theorem 2.

Using the solution for the electric field one can see that

$$-\frac{1}{2} \int j^* E(x) dx = -\frac{\pi\omega\mu}{kc^2} j_0^2 = \frac{\pi|\mu|}{cn} j_0^2 > 0. \tag{19}$$

The refractive index n obeys the usual equation $n = ck/\omega$. It is easy to show that $S_x = \pi|\mu| j_0^2/2cn > 0$ at $x > 0$ and that $S_x = -\pi|\mu| j_0^2/2cn < 0$ at $x < 0$.

Now we analyze solution of SK. It reads

$$E(x,t) = -\frac{2\pi\omega}{n_{-}\kappa c^2} \mu j_0 e^{i(n_{-}\kappa|x| - \omega t)}. \tag{20}$$

With $n_- = n'_- + in''_-$ this equation takes the form

$$E(x,t) = -\frac{2\pi\omega}{n_-\kappa c^2}\mu j_0 e^{(in'_- - n''_-)\kappa|x| - i\omega t}. \tag{21}$$

It follows then, that the solution written in this form dictates n''_- to be positive. This result also follows from Theorem 2 for the LHM because n'_- is negative.

Using expression Eq.(20) one gets

$$-\frac{1}{2}\int j^* E(x)dx = \frac{\pi\mu}{cn_-}j_0^2 = \frac{\pi|\mu|}{cn}j_0^2 > 0. \tag{22}$$

Thus, we see that the descriptions of this problem for the LHM in terms of n and n_- are identical. The SK's solution Eq.(21) can be obtained from Eq.(18) by changing $n \to -n_-$.

Moreover, even the RM can be described in terms of negative refractive index n_-. Suppose, that our sheet of current is embedded into a RM. Then the correct solution for the electric field can be written in the form

$$E(x,t) = \frac{2\pi\omega}{n_-\kappa c^2}\mu j_0 e^{i(-n_-\kappa|x| - \omega t)}. \tag{23}$$

Note that this time n''_- should be negative, that also supports our theorem because we consider the RM with negative n'_-.

The above examples confirm our point of view that the sign of n' is not the criterion which makes a difference between the LHM and RM. We think that these examples reflect the general case and the reason is that any electrodynamics problem can be formulated in terms of ϵ and μ, so that the question of the sign of n does not appear.

Since the mathematical description with both n and n_- is the same, we doubt that there is a way to find the sign of n either from experimental or from computational data.

Smith *et al.*[8] attempted to find the sign of n from the reflection and transmission. Transmission t and reflection r coefficients for waves normally incident on a LHM slab with the width d can be written in a form

$$t^{-1} = \left[\cos(n\kappa d) - \frac{i}{2}(\mu + \epsilon)\frac{\sin(n\kappa d)}{n}\right]e^{i\kappa d} \tag{24}$$

$$r = -\frac{i}{2}te^{i\kappa d}(\mu - \epsilon)\frac{\sin(n\kappa d)}{n}. \tag{25}$$

These equations are invariant with respect to change $n \to -n_-$. However the authors of [8] solve them with respect to n',n'' and impose the condition $n'' > 0$. Then, in agreement

with Theorem 2, they find that in the LHM $n' < 0$. They claim that the condition $n'' > 0$ is valid for any passive material. From our point of view the conditions Im$\epsilon > 0$ and Im$\mu > 0$ define a passive material, and we are not aware of any theorem about the sign of n''.

FERMAT'S PRINCIPLE FOR THE LHM

Fermat's principle for the LHM has been recently formulated by Veselago [13] in terms of negative refractive index. Here we show how to do it keeping the refractive index positive for both LHM and RM. A simple generalization of the Maupertuis principle for the particles reads

$$\delta \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{k} \cdot d\mathbf{l} = 0, \tag{26}$$

where $d\mathbf{l}$ is a vector in the direction of propagation of photons which is parallel to the GV or to the Poynting vector. In the RM $\mathbf{k} \cdot d\mathbf{l} = (\omega n/c)dl$, while in the LHM $\mathbf{k} \cdot d\mathbf{l} = -(\omega n/c)dl$. Equation(26) describes the propagation of rays. Following Veselago[13] we apply this principle to refraction of the waves at the RM-RM and RM-LHM interfaces. Consider a point A with coordinates $(0, -a)$ in a RM with the refractive index n and a point D with coordinates (y^*, z^*) in the material with the positive refractive index n_1 (Fig. 2). We consider two cases: first - when the media at $z > 0$ is a RM, second - when it is a LHM described by a positive refractive index n_1. Using Fermat's principle we find the path of the ray of light coming from A to D. Equation (26) in the case of RM-RM interface has a form

$$\frac{ny}{\sqrt{a^2 + y^2}} + \frac{n_1(y - y^*)}{\sqrt{(z^*)^2 + (y - y^*)^2}} = 0. \tag{27}$$

This equation gives the usual Snell's law $n \sin\theta_i = n_1 \sin\theta_r$ because $\sin\theta_i = y/\sqrt{a^2 + y^2}$, $\sin\theta_r = (y^* - y)/\sqrt{(z^*)^2 + (y^* - y)^2}$. The propagation of rays in this case is shown at Fig. 2 by the dashed line ACD.

In the case of RM-LHM interface the Fermat's principle Eq.(26) gives

$$\frac{ny}{\sqrt{a^2 + y^2}} - \frac{n_1(y - y^*)}{\sqrt{(z^*)^2 + (y - y^*)^2}} = 0, \tag{28}$$

which is the anomalous Snell's law $n \sin\theta_i = -n_1 \sin\theta_r$ providing the negative refraction. The propagation of rays is shown by the solid line ABD at Fig. 2. Note, that $n = n_1$ is a special case. Under this condition there is a focal point inside the LHM with coordinates

$(0, a)$. One can see from Eq.(28) that at $n = n_1$ all rays go through this point. The optical length of the path, which can be defined as

$$\frac{c}{\omega}\int_A^D \mathbf{k}\cdot d\mathbf{l}, \tag{29}$$

is zero for any ray going from A to the focal point. The focal point is absent for $n \neq n_1$ [1].

CONCLUSIONS

1. We have proven two theorems based upon fundamental thermodynamic inequalities that show the difference between the LHM and the RM. The first one states that the GV is negative in the LHM and positive in the RM. If the losses are small, one could get all Veselago's results from this theorem without using the first order Maxwell's equations. The second theorem claims that the product $n'n''$ is positive in the RM and negative in the LHM.

2. Both theorems are invariant with respect to change of the sign of refractive index.

3. We have shown that the introduction of negative refractive index is possible in both RM and LHM. However it may be misleading because some traditional electrodynamics expressions including Eq.(9) become incorrect if n is negative. That is the reason why some authors[5, 7] claim that GV in the LHM is positive.

4. We have generalized the Fermat's principle for the LHM using positive value of n and have shown that it leads to a negative refraction at the RM-LHM interface.

5. Thus we do not see any reason to ascribe a negative refractive index to the LHM.

The work has been funded by the NSF grant DMR-0102964.

[1] V. G. Veselago, Sov. Phys.-Solid State **8**, 2854 (1967).

[2] V. G. Veselago, Sov. Phys. Uspekhi **10**, 509 (1968).

[3] R. A. Shelby, D. R. Smith, and S. Schultz, Science **292**, 77 (2001).

[4] H. Kosaka, T. Kawashima, A. Tomita, M. Notomi, T. Tamamura, T. Sato, and S. Kawakami, Phys. Rev. B **58**, 10096 (1998).

[5] D. R. Smith and N. E. Kroll, Phys. Rev. Lett. **85**, 2933 (2000).

[6] N. Garcia and M. Nieto-Vesperinas, Phys. Rev. Lett. **88**, 207403 (2002).

[7] P. M. Valanju, R. M. Walser, and A. P. Valanju, Phys. Rev. Lett. **88**, 187401 (2002).

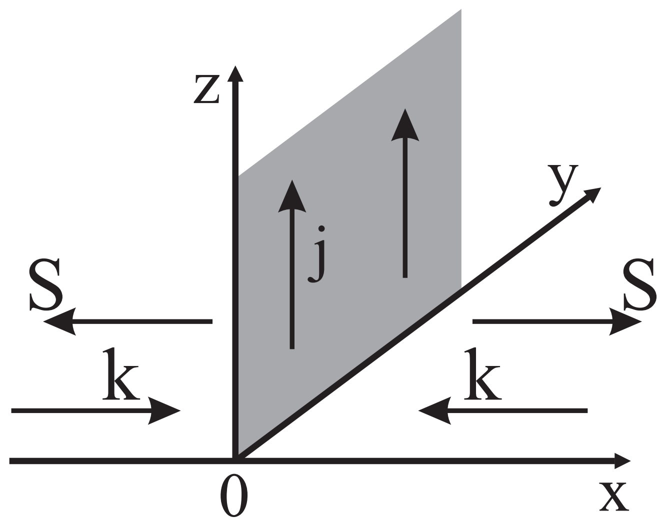

FIG. 1: An infinite sheet of current in the $y-z$ plane, embedded into the LHM. **S**, **k** and **j** show the directions of the Poynting vector, the wave vector and the current respectively.

[8] D. R. Smith, S. Schultz, P. Markos, and C. M. Soukoulis, Phys. Rev. B **65**, 195104 (2002).

[9] J. B. Pendry, Phys. Rev. Lett. **85**, 3966 (2000).

[10] P. Markos and C. M. Soukoulis, Phys. Rev. B **65**, 033401 (2001).

[11] D. L. Landau, E. M. Lifshitz, and L. P. Pitaevskii, *Electrodynamics of Continuous Media* (Butterworth Heinemann, 1984), p.284.

[12] L. Brillouin, *Wave propagation and group velocity* (Academic Press, New York and London, 1960), p. 47.

[13] V. G. Veselago, cond-mat/0203451.

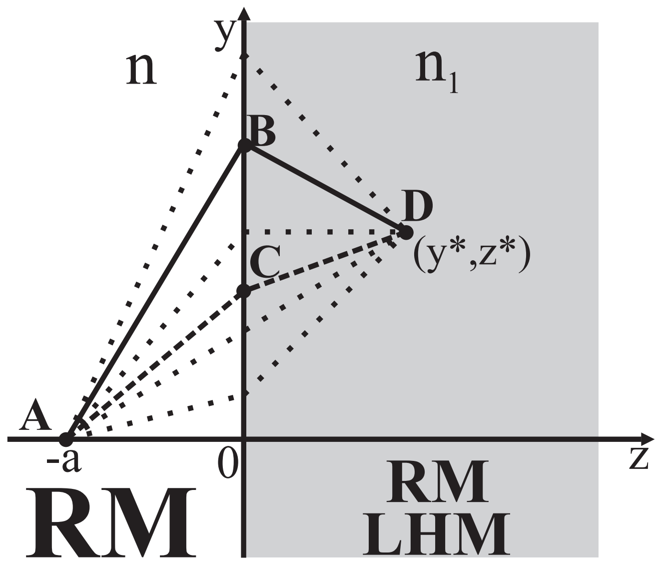

FIG. 2: Different paths of rays traveling from point A to point D. If the medium at $z > 0$ is a LHM, the path ABD obeys the Fermat's principle. It shows negative refraction. In the case when both media are RM the path ACD, which has a positive refraction, obeys the Fermat's principle.